Solutions to a Challenging Problem on Integers

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Abstract -- **Solutions to the challenging problem: Find an *n*-digit number N_n whose last digit if shifted to the extreme left so that the modified number N_m is mN_n where *m* is a multiplier (5/4), is provided by Surjit Singh. However, we propose here two more alternative solutions for the same. We generalize this problem and provide its solutions. A simple multiplication method for a special case is discussed in detail. A number of illustrative examples are included.**

Keywords: *Integers, number theory,*

I. INTRODUCTION

The problem: Let an *n*-digit number be

$$N_n = d_1 d_2 \ldots d_{n-1} d_n = A_{n-1} d_n, \quad (1)$$

where

$$A_{n-1} = d_1 d_2 \ldots d_{n-1}. \quad (2)$$

Let d_n is shifted to extreme left so that the modified number

$$N_m = d_n d_1 d_2 \ldots d_{n-1} = d_n A_{n-1} = mN_n \quad (3)$$

where *m* is a multiplier (5/4). The problem and its solution are available on U-Tube [1]. In this paper, we suggest two alternative methods in Section II. We generalize the problem and give its solutions in Section III. We also discuss one special case when *m* is an integer greater than 1. Section IV gives the conclusion.

II. TWO ALTERNATIVE SOLUTIONS

Solution 1: In this solution we find $d_1, d_2, \ldots d_n$ in sequence.

Step 1:
From (1)-(3), we observe that the first *n*–1 digits of N_n are the same as those of the last *n*–1 digits of N_m.
Given that

$$N_m = N_n(5/4) \quad (4)$$
$$\rightarrow N_n = N_m \times 0.8.$$

Let us assume $d_1 = 1$. Since the number of digits remains the same, there should not be any overflow at the last step when we execute (3).

Since the factor (5/4) is not equal to but close to and greater than 1, d_n will be 2. Thus, $d_1 d_2 = 21$.

Step 2:
Since 21×0.8 = 16.8. It should be an integer. Therefore, $d_1 d_2 = A_2 = 17$. Subsequent digits are obtained using (4) as follows.
$2A_2 \times 0.8 = 173.6 = A_2 3.6 A_2 3 = A_3,$
$2A_3 \times 0.8 = 1738.4 = A_4 8.4\ A_3 8 = A_4.$

Proceeding further, we get
$2A_{15} \times 0.8 = 173913043478260 = A_{16}.$

Since there is no overflow, we take 60 as the next two digits.
$2A_{16} \times 0.8 = 173913043478258608\ \ A_{16} 8 = A_{17}.$
To conserve the space, we give the last two steps.
$2A_{19} \times 0.8 = 1739130434782608695651.2 = A_{19} 1.2\ A_{19} 1 = A_{20}$
$2A_{20} \times 0.8 = 1739130434782608695652.$

We note that there is no overflow in the last operation, also, we got $d_1 = 1$ and $d_n = 2$ with which we started. The process, therefore, ends here. Thus, we arrive at

$$N_{22} = 1739130434782608695652.$$

Solution 2
In this method we find $d_n, d_{n-1}, \ldots, d_1$ in sequence.
The $d_1 = 1$ and $d_n = 2$ as argued in solution 1 above. From the given condition, N_n should be divisible by 4. Hence, the last two digits of N_n should also be divisible by 4. Therefore, $d_{n-1} d_n$ can be one of 12,32,52,72,92. After dividing N_n by 4, the result is to be multiplied by 5. This implies that d_{n-1} can be either 0 or 5, depending upon $d_{n-1}2/4$ (integer)is even or odd, respectively. Therefore, d_{n-1} has to be 5. Hence $d_{n-1} d_n = 52$.

The values of $(d_{n-2}52) \times (1.25)$ for integer values of d_{n-2} are given in Table 1. We notice that $d_{n-2} d_{n-1}$ in two columns match. It indicates that d_{n-3} is 6.

TABLE-1

$D3 = d_{n-2}52$	$B3 = (D3)(1.25)$
052	0**65**$=0$**A_2**
152	190
252	315
352	440
452	565
552	690
652=**A_2**2	

We follow the similar procedure for evaluating d_{n-4} and prepare Table 2. Put integer values starting from 0 till $A_3 = d_{n-3}d_{n-2}d_{n-1}$ = 565 appear in two columns. It indicates that d_{n-4} is 5.

TABLE 2

D4=d_{n-3}652	B4 =(D4)(1.25)
0652	815
1652	2065
2652	3315
3652	4**565 =**4**A**$_3$
4652	5815
5652=$A_3$2 →	

Table 3 shows that $A_4 = d_{n-4}d_{n-3}d_{n-2}d_{n-1}$ values match in the two columns. It indicates d_{n-5} has to be 9.

TABLE 3

D5=$d_5$5652	B5 =(D5)(1.25)
05652	7065
15652	1**9565=**1**A**$_4$
95652 = **A**$_4$2 →	

Now we formulate the procedure to get (i+1)th digit when last i digits are known. Prepare a Table 4 as shown below. Put the integer values starting from 0 for d_{i+1} digit till Bi matches withA_i =

TABLE 4

D(i+1)=d_{i+1}Di	B(i+1) =D(i+1)(1.25)
0Di	.
1Di	.
.	.
.	.
	..$b_{i+1}$$\boldsymbol{A_i}$
$\boldsymbol{A_i}$2 →	

$d_{i+1\ldots}d_2$. Then d_{i+1} is the next digit. One need not proceed further. Note that $b_{i+1}A_i$may be in any row.

Following the above procedure, we determined A_{19}2 = 391304782608695652. For further digits, we are giving the results in Tables 5 and 6. Table 5 indicates that d_{21} is 7.

TABLE 5

D21=d_{21}**39130478260869565**2	B21=(D21)(1.25)
0391304782608695652	048913043478260869565
1391304782608695652	223913043478260869565
2391304782608695652	298913043478260869565
3391304782608695652	423913043478260869565
4391304782608695652	548913043478260869565
5391304782608695652	6**73913043478260869565 =6*A*$_{19}$**
7391304782608695652 =A_{19}2 →	

TABLE 6

D22=d_{22}**739130434782608695652**2	B_{22}=(D22)(1.25)
0739130434782608695652	923913043478260869565
1739130434782608695652 = $\boldsymbol{A_{21}}$2	2**173913043478260869565 =2*A*$_{21}$**

Table 6 shows that d_1 is 1. Since we come to d_1= 1 and d_n= 2, we stop here. Thus, the desired number is a N_{22} = 1739130434782608695652.

III. GENERALIZED PROBLEM

Find all possible n-digit numbers $N_n = d_1d_2\ldots d_{n-2}d_{n-1}d_n$ such that

$$N_m = mN_n, m > 1$$
$$= \begin{cases} d_n d_1 d_2 \ldots d_{n-1}, m \text{ non} - \text{intger}, d_n \neq 0 \\ d_{n-1} d_n d_1 d_2 \ldots d_{n-2}0, m\text{ineger}, d_n = 0, \end{cases}$$

where m is a multiplier.

Division Method (m is a non-integer)

It is given that

$$N_m = mN_n.$$

Let m= (α/β), α and β are integers, α ≠ β.

Using (1), (2) and (3), we get

$$(\alpha/\beta)(A_{n-1}d_1) = (d_1A_{n-1}) \rightarrow \alpha(A_{n-1}d_1) = \beta(d_1 a) = (\beta - 1)99\ldots\ldots.99(10 - \alpha)d_1 \tag{5}$$

$$\rightarrow \frac{A_{n-1}}{d_1} = \frac{(\beta - 1)99\ldots\ldots.99(10 - \alpha)}{(10\alpha - \beta)\ = p_1p_{2.}}, \tag{6}$$

where p_1 are p_2 the two integer factors of (10α-β) such that p_1 is a single digit number. Identify

$$d_1 = p_1. \qquad (7)$$

and

After obtaining using (8), (1) gives

$$A_{n-1} = \frac{(\beta - 1)99 \ldots \ldots .99(10 - \alpha)}{p_2} = \frac{\lambda}{p_2}. \qquad (8)$$

$N_n = An_{-1}d_1$.

From (8), we note the following facts.

1. Since A_{n-1} is an integer, right-hand side of (8) should be an integer, i.e., λ should be an integer multiple of p_2.

2. If (10α-β) does not have two integer factors, then p_1 should be considered as 1. Thus, d_1 is 1.

3. $\alpha \leq 10$ and$1 \leq \beta \leq 10$.

4. When β = 1, then d_1 is 0.

5. If β =10 and α = 1,

$$A_{n-1} = \frac{99 \ldots \ldots .99}{0} = \infty.$$

Thus, there is no solution.
If β =1 and α = 10,

$$A_{n-1} = \frac{099 \ldots \ldots .990}{11} = \overline{09}.$$

Hence there is no solution.

6. If α = 1

$$A_{n-1} = \frac{(\beta - 1)99 \ldots \ldots .999}{10 - \beta} = \frac{N}{D}.$$

a) N is (odd). For β = 2,6, and 8, D is even. Therefore, A_{n-1} being a ratio of odd/even, is a non-integer.
b) If β = 5, A_{n-1} will not be an integer, as it requires 0 or 5 as the last digit in N.
c) A_{n-1} a non-integer for β = 7,9. Therefore, there is no solution.

Thus, only β = 3,4 will yield proper $N_{n.}$

(7) If is odd and p_2 is even, then λ/p_2 will not be an integer. In such cases, there will be no solution.

8. If we take a greater number of 9's in (8) than necessary, it will repeat the same cycle and increase the number of digits, which is not permissible.
9. The number of solutions will depend upon the number of valid choices for $p_{1,2}$, *i.e.*, the integer factors of (10α–β)
10. If both α and β are multiplied by *k*, N_n remains the same.
11. If $A_{n-1} = \frac{(a99\ldots\ldots99)}{p_2}$ and a^9 is divisible by p_2, we do not require any additional 9 to get as an integer. However, it does not give the correct solution.
12. If the last digit in N_n is 0, then one has to bring the last two digits to extreme left. When a zero is multiplied by any number, it gives a zero. Therefore, a 0 is required to be added at the end to get correct N_m. However, it will increase the number of digits by 1.

Multiplication method when β = 1

We give a multiplication method using the following recursive formula.

$$d_{i-1} = md_{i} + c_{i+1}, \quad i = n, n-1, \ldots, 2, \quad c_{i+1} = 0 \qquad (9)$$

If $m \times d_2 + c_3$ has a carry, then the number of digits in N_m becomes $n+1$. In this case, we will not have the correct N_m. The method is explained through examples 14-16.

Examples (Refer to Table 7)

In Example 1

$$\frac{A_{n-1}}{d_1} = \frac{399 \ldots \ldots .995}{2 \times 23}. \qquad (10)$$

The only choice for d_1 is 2. Then

$$A_{n-1} = \frac{399 \ldots \ldots .995}{23} = 17391304347826086956 5. \qquad (11)$$

and $N_{22} = d_1 = 1739130434782608695652$.

This N_{22} is the unique value for 22 digits. The result is given in Example 1 of Table 7. If we take more numbers of 9s, it will result at the end of second cycle into 17391304347826086956521739130434782608695652. It will go on in cyclic order.

Similarly, for other values of α and β, the results are given in examples 2 to 12 based on division

1	2	3	4
Example	α, β	$A_{n-1} = \frac{N}{D}$	N_n
			I. Division method
			Case A: m is a non-integer (α > β)
1.	5, 4	$\frac{399\ldots\ldots995}{23}$	1739130434782608695652
2.	6, 5	$\frac{499\ldots\ldots994}{1\times 55 \text{ or } 5\ \times 11}$	55 does not give; but 11 gives 45
3.	10, 3	$\frac{299\ldots990}{97}$	LSDs of N and D are 0 and 7. N/D is non-integer.
4.	10, 2	$\frac{1999\ldots\ldots90}{49 \text{ or } 14}$	49 does not give integer value; but 14 gives142857.After cancelling the common factor 2 in α and β, it also gives the same solution as in Example 7.
5.	3, 1	$\frac{0999\ldots\ldots997}{29}$	0344827586206896551724137931(28 digits)
6.	4,1	$\frac{0999\ldots\ldots996}{39 \text{ or } 13}$	Choosing $p_2 = 39$, $N_6 = 025641$ and Choosing $p_2 = 13$, $N_6 = 076923$.
7.	5, 1	$\frac{0999\ldots\ldots995}{49 \text{ or } 7}$	49 does not give, but 7 gives 0142857.
8.	1, 3	$\frac{299..99}{7}$	428571
9.	1, 4	$\frac{399..99}{3} = 13$	HereN_3 = 132. But 213 is not the correct vale of N_m because 132/4 = 33. Even though $A_{n-1}$is an integer, we do not get the correct N_n.
10.	1, 7	$\frac{699..99}{3} = 23$	No solution, as explained in Example 9.
11.	6, 7	$\frac{699..994}{53}$	$1\overline{132075471}$, No solution.
12.	8, 9	$\frac{899..992}{71}$	12676056303807464788732394366197183098591521.

TABLE 7

		II. Multiplication method, β= 1
13.	3, 1	0344827586206896551724137931 (basic N_n) Derived7N_n values are 1034482758620689655172413793 3103448275862068965517241379 1379310344827586206896551724 1724137931034482758620689655 0689655172413793103448275862 2068965517241379310344827586 2758620689655172413793103448 Additional N_n with 0 at the end and corresponding N_m 3448275862068965517241379310 10344827586206896551724137930 6896551724137931034482758620 20689655172413793103448275860
14.	4, 1	Choosing p_2= 13, N_6 =076923. Choosing p_2 = 39, N_6 = 025641. The other possible values from the circlesare N_6 = 102564 from Fig. 2(a)and N_6 = 076923 and 230769 from Figure 2(b) Also, those which have 0 at the end are N_6 = 769230 which gives N_m = 3076920 (7 digits) N_6 =256410 which gives N_m = 1025640 (7 digits)
15.	5, 1	020408163265306122448979 591836734693877551 040816326530612244897959183673469387755102 061224489795918367346938775510204081632653 081632653061224489795918367346938775510204 102040816326530612244897959183673469387755 122448979591836734693877551020408163265306 163265306122448979591836734693877551020408 183673469387755102040816326530612244897959.
16	12, 1	008403361354437815726050420168067226390756302521

Examples 13-16 are based on the multiplication method.

Example 14

Let d_n be 1 and m be 3. Using (19), we get
$1\times 3 = 03$. Thus, $d_{n-1.} = 3$ and $c_{n-1} = 0$.
$3\times 3 + 0 = 9$. Thus, $d_{n-2.} = 9$ and $c_{n-2} = 0$.
$9\times 3 + 0 = 27$. Thus, $d_{n-3.} = 7$ and $c_{n-3} = 2$.
$7 \times 3 + 2 = 23$. Thus, $d_{n-4.} = 3$ and $c_{n-4} = 2$.

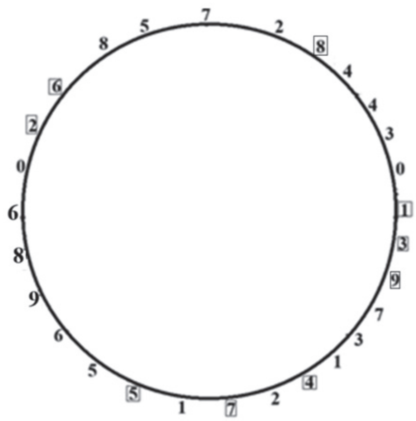


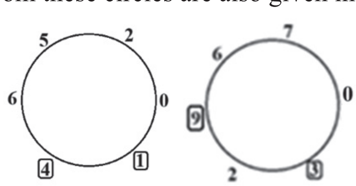
Figure 1. Pictorial representation for example.

Proceeding in this manner, we get
N = 1034482758620689655172413793l.

The result of Example 13 is pictorially shown in Figure 1. Note that the numbers 1,3,9,4,7,5,2,6,8 enclosed in rectangles are generated without any carry from the previous number. Thus, not only series starting from 1, but also from these numbers give correct N_n. These 9 values of N_n are shown in the Table 7.

Two more N_n which have 0 at the end are obtained from the circle shown in Figure 1. From these two more N_m are obtained as follows. Move x0where x is 1 and 2 to the left of $d_{1.}$ Since 0 when multiplied with any number gives 0, add a 0 at the end. Note that these N_m require n+1 digits.

The circle for Example 14 is shown in Fig. 2 (a). It was also obtained by division method for example6choosing $p_2 = 39$ and 13. The latter is shown in Figure 2(b). Other possible solutions are obtained from these circles are also given in Table 7.



(a) (b)

Figure 2. Circles for Example 15 (m = 4).

Solutions for m = 5 are given in Example 15 of Table 7. We have not included the results which have 0 at the end, and the circle. Example 16 is for $\alpha = 12 > 10$.
For examples 6 and 13, a 0 is present in N_n and $N_{m.}$ Hence it can be removed from both.

Comparison between division and multiplication methods

1. The multiplication method gives equal or larger values of N_n than that obtained by the division method.
2. The multiplication method is much simpler than the division method.
3. The multiplier $m \leq 10$ in the division method while it is any integer in multiplication method.
4. The number of solutions in division method depends upon the valid values of p_2. A greater number of solutions are possible in the multiplication method, including those which has n+1 digits.

Solution 2: Inverted long division method
We explain the method with two examples.

1) Consider again Example 1. From (8),

$$A_{n-1} = \frac{39\ldots\ldots.95}{23}.$$

This division is executed in the inverse manner as follows. Refer to Fig. 4. Every integer 1 to 9 appears once as the last digit in the multiples of 23. For example, the last digit 5 appears in the multiples of 23 as 5×23=115. Therefore, the remainder of the previous step has to be 11. Now 9 comes down. Obviously, the last digit has to be 8 which comes in the multiple of 23 as 6×23 = 138. Thus, the last two digits of N_n are 65. Now another 9 comes down. Obviously, the last digit has to be 5 which appears in the multiple of 23 as 5×23 = 115. Therefore, the last three digits of N are 565. Again another 9 comes down. And the last digit has to be 7 which appears in the table of 23 as 207 = 9×23. Thus N_4 = 9565.

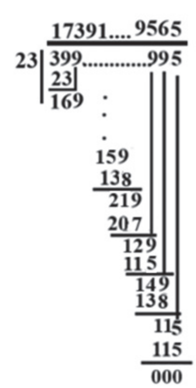


Figure 3. Inverted long division by 23.

Proceeding in this way, we go up till we come across the number 39. The process stops here. Finally, we get . Then $N_{22} = A_{21}\, p_1$ =17391304347826086956522.

2) Consider again Example 6. Here

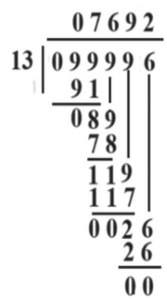


Figure 4. Inverted long division by 13.

Following the inverse division method, we find A = 07692 as shown in Fig. 4. Then $N = Ap_1$ = 027692.

IV. CONCLUSION

We have given two alternative solutions to the problem: Find an n-digit number N whose last digit if shifted to the extreme left so that the modified number N_m is (5/4)N without increasing the number of digits. Then we have generalized this problem and suggested two solutions for it. Finally, we have discussed the special case when multiplier is an integer > 1. In this case, we get the various values of N and, therefore, we can choose the maximum or minimum value.

REFERENCE

[1] Surjit Singh, The problem and the solution, Surjitcode@gmail.com

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