Circuit Theorems – Scope and Limitations

(Part II: Maximum power delivery and superposition theorems)

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Abstract - **This is the second part of three parts of the above titled paper. This part will deal with maximum power transfer theorem, maximum power delivery theorem and true superposition theorem.**

Keywords: Superposition theorem, maximum power transfer theorem, maximum power delivery theorem, True superposition theorem.

I. INTRODUCTION

THE MAXIMUM power transfer theorem has received some attention by several authors [1] - [9]. References [1] - [3] deal with the maximum power transfer in multi-port networks. Hence, their description will be out of place as we will address the power transfer from a dc 2-terminal network *n* to another 2-terminal network *N*. Most of the books deal with the maximum power transferred to a variable load resistance. Very few books, for example [10] considered the maximum power delivered to a fixed load resistance. Here, all the 3 possible cases of power transfer to a resistive load [6] are considered in Section II.

However, all these are restricted to a resistance only that absorbs the power. Narayanan [7] has given a generalized form of maximum power transfer in which the element absorbing power need not be just a resistor. The theorem can be stated as: *the maximum power is transferred from a 2-terminal network n to another 2-terminal network N when voltage across N equals half the open circuit voltage of n or current through N becomes half of the short circuit current of n.* Narayanan Proof of the theorem [7] and an alternative proof [9] are given in Section III.

Leach [11] found that 20 introductory books on circuit analysis [12-31] either state or imply that principle of superposition (POS) on dependent sources is not allowed, which, he contended, is a misconception. He finally concluded that POS can be applied to such networks also, through a formal proof followed by several examples. In this paper, we give a general condition on the circuits to which POS cannot be applied.

A simple, but convincing proof [32] is provided in Section IV. The results are verified by the matrix method [33]. Finally, it is shown that the matrix method is more efficient [33-34].

II. MAXIMUM POWER TRANSFER THEOREM

It is desired to maximize the power in a load resistance R_L when only one resistance is variable. One of the steps in evaluating the power transferred to R_L is to find the Thevenin Equivalent across R_L. This is shown in Figure 16 where R_T and V_T are the Thevenin Resistance and Thevenin Voltage, respectively.

The power delivered to R_L is given by

$$P = \left(\frac{V_T}{R_T+R_L}\right)^2 R_L = \frac{V_T^{\,2} R_L}{R_T^{\,2} + 2R_T R_L + R_L^{\,2}} = \frac{N}{D}. \tag{37}$$

The power will be maximum/minimum with respect to a variable x, when

$$\frac{dP}{dx} = \frac{d}{dx}\left(\frac{N}{D}\right) = \frac{D\frac{dN}{dx} - N\frac{dD}{dx}}{D^2} = 0 \tag{38}$$

i.e., when

$$\frac{N}{D} = \frac{dN/dx}{dD/dx}. \tag{39}$$

Cases of power transfer:

Since there are three variables, R_L, R_T and V_T, the following are the possible combinations.

A) R_L alone is variable

B) R_T alone is variable

C) V_T alone is variable

D) R_L and R_T both are variable

E) V_T and R_T both are variable

F) V_T and R_L both are variable

G) V_T, R_T and R_L are variables

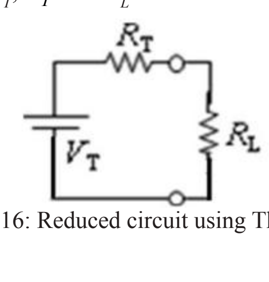


Figure 16: Reduced circuit using Thevenin theorem.

Since one and only one resistance is allowed to vary, and V_T and R_T do not depend upon R_L, cases D, E, F and G are not possible. If V_T is a variable as a function of some resistance, then R_T being the ratio of open circuit voltage (V_T) and short circuit current also varies with R. Thus case C does not exist. The remaining three cases will be considered now.

Case *(i)*: R_L alone is variable

The power will be maximum/minimum, using eqn. (39), when

$$\frac{N}{D}=\frac{dN/dR_L}{dD/dR_L} \tag{40}$$

i.e., when

$$\frac{V_T{}^2 R_L}{R_T{}^2+2R_TR_L+R_L{}^2}=\frac{V_T{}^2}{2R_T+2R_L}.$$

This leads to the condition

$$R_L=R_T.$$

This result is explicitly covered in the maximum power transfer theorem which states that the power transfered in R_L is maximum when R_L equals R_T.

Case *(ii)*: When R_T alone is variable

The power P will be maximum/minimum when, from eqn. (39),

$$\frac{d}{dR_T}D=0$$
$$\Rightarrow 2R_T+2R_L=0$$

i.e., when $R_T = -R_L$. Note from eqn. (37) that P decreases as R_T increases. Hence if R is to be a non-negative resistor then $R_T = 0$ will give the maximum power. Thus the conditions for maximum power in cases (i) and (ii) are different.

Example 1: Consider the circuit shown in Figure 17(a). This comes under Case 1.

The above procedure indicates that the power transferred to the combination of load resistance R_L and R_o will be maximum when their parallel equivalent resistance equals R. But this condition will not necessarily give the maximum power in R_L. Drawing the Thevenin equivalent circuit as shown in Fig. 17(b), and applying the above theory, we see that the power is maximum when $R_L = RR_o/(R+R_o)$.

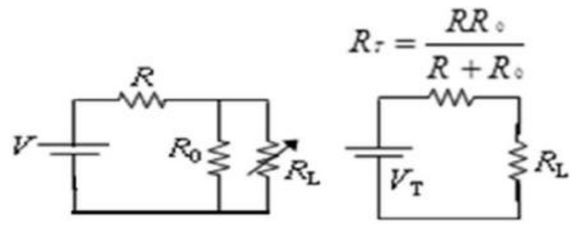


Figure 17. (*a*) Circuit for Example 1, (*b*) Reduced circuit.

Example 2: Consider the circuit shown in Figure 18.

This belongs to Case (ii). Here the power delivered in R_L will be maximum when equivalent resistance of the parallel combination of R and R_o equals 0, i.e., when $R = 0$.

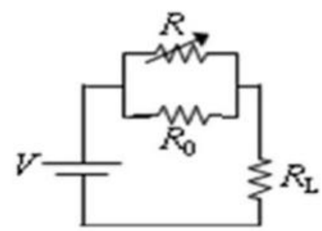


Figure 18. Circuit for Example 2.

Case *(iii)*: When both R_T *and* V_T *are* variable

In Figure 16 if both V_T and R_T are the functions of some resistance R, then the condition for maximum/ minimum power, from eqn. (40), is

$$\frac{V_T{}^2 R_L}{R_T{}^2+2R_TR_L+R_L{}^2}=\frac{R_L 2V_T\frac{d}{dR}V_T}{\left(2R_T+2R_L\right)\frac{d}{dR}R_T}$$
$$\Rightarrow \frac{V_T}{R_T+R_L}=\frac{\frac{d}{dR}V_T}{\frac{d}{dR}R_T}. \tag{41}$$

This will give the value of R for maximum/minimum power transfer to R_L. Note that eqn. (41) is valid when both V_T and R_T are expressed as polynomials in R.

Example 3: Consider the circuit shown in Fig.19. First we determine its Thevenin equivalent. The open circuit voltage V_T and short circuit current I_S are determined using loop analysis as follows. Applying KVL to the circuit of Fig.20 when terminals XY are open, we get

$$\begin{bmatrix}3+R & -3\\ -3 & 9\end{bmatrix}\begin{bmatrix}I_1\\ I_2\end{bmatrix}=\begin{bmatrix}10\\ 0\end{bmatrix}.$$

By Cramer rule,

$$I_1 = \frac{\begin{vmatrix} 10 & -3 \\ 0 & 9 \end{vmatrix}}{\begin{vmatrix} 3+R & -3 \\ -3 & 9 \end{vmatrix}} = \frac{10}{R+2} \quad (42)$$

and

$$I_2 = \frac{\begin{vmatrix} 3+R & 10 \\ -3 & 0 \end{vmatrix}}{\begin{vmatrix} 3+R & -3 \\ -3 & 9 \end{vmatrix}} = \frac{10}{3(R+2)}. \quad (43)$$

Thus

$$V_T = RI_1 + 3I_2 = \frac{10(R+1)}{R+2}. \quad (44)$$

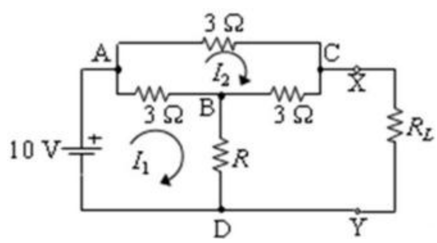


Figure 19. Circuit for the Example 3.

Again applying KVL to the circuit of Fig.19 when terminals XY are shorted, and I_S is the clockwise loop current in loop DBCY, we get

$$\begin{bmatrix} 3+R & -3 & -R \\ -3 & 9 & -3 \\ -R & -3 & R+3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_S \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}.$$

By Cramer Rule,

$$I_S = \frac{\begin{vmatrix} 3+R & -3 & 10 \\ -3 & 9 & 0 \\ -R & -3 & 0 \end{vmatrix}}{\begin{vmatrix} 3+R & -3 & -R \\ -3 & 9 & -3 \\ -R & -3 & 3+R \end{vmatrix}} = \frac{10(R+1)}{2R+3}.$$

Now

$$R_T = \frac{V_T}{I_S} = \frac{(2R+3)}{R+2}. \quad (45)$$

Note from eqns. (44) and (45) that both V_T and R_T are the functions of R.

Now P is given by

$$P = \left[\frac{\frac{10}{R+2}(R+1)}{\frac{2R+3}{R+2} + R_L} \right]^2 R_L$$

$$= \frac{[10(R+1)]^2 R_L}{[(R_L+2)R + (2R_L+3)]^2}$$

$$= \frac{[V_T']^2 R_L}{[R_T' + (2R_L+3)]^2} \quad (46)$$

where

$$V_T' = 10\,(R+1),\ R_T' = (R_L+2)R;$$

both are the polynomials in R. The condition for maximum/minimum power, from eqn. (39), is

$$\frac{V_T'}{R_T' + R_L} = \frac{\frac{d}{dR} V_T'}{\frac{d}{dR} R_T'}$$

$$\frac{10(R+1)}{(R_L+2)R + (2R_L+3)} = \frac{\frac{d}{dR}\{10(R+1)\}}{\frac{d}{dR}\{(R_L+2)R\}}$$

$$= \frac{10}{R_L+2}.$$

This does not yield a solution for R. It implies that P is a monotonic function. From eqn. (46), we see that P increases as R increases. Hence it will be minimum when $R = 0$ and maximum when $R = \infty$. Thus the maximum value is

$$\frac{100R_L}{(R_L+2)^2} = 11.11 \text{ when } R_L = 1.$$

Example 4: Consider the circuit shown in Figure 20(a). It has one controlled source. We shall first find the Thevenin equivalent for the circuit. Then the condition for maximum power transferred can be obtained from eqns. (40) or (41). We shall find out the Thevenin equivalent by two alternative methods: Applying an external voltage source and finding the current supplied by this source by loop method, and by applying an external current source and again solving by the loop method.

External voltage source and loop analysis

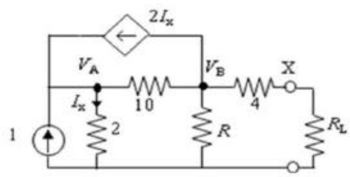
Let a voltage source be connected across terminals XY as shown in Figure 21(b). Loop currents are also shown in the figure. Note that

$$I_X = 1 - I_1. \quad (47)$$

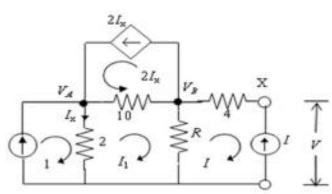
After converting both the current sources into voltage sources, the circuit becomes as shown in Figure 20(c). KVL gives

$$\begin{bmatrix} R+12 & -R \\ -R & R+4 \end{bmatrix}\begin{bmatrix} I_1 \\ I \end{bmatrix} = \begin{bmatrix} 2-20I_X \\ -V \end{bmatrix}.$$

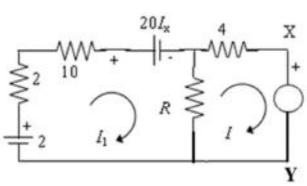
Substituting for I_x from (47),



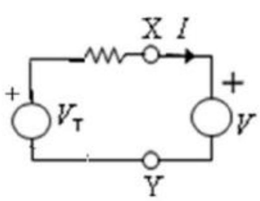
(a)



(b)



(c)



(d)

Figure 20.*(a)* Circuit for example 4.
(b) A voltage source is applied across XY
(c) Current sources converted into voltage sources
(d) Thevenin Equivalent.

$$\begin{bmatrix} R+12 & -R \\ -R & R+4 \end{bmatrix}\begin{bmatrix} I_1 \\ I \end{bmatrix} = \begin{bmatrix} 2-20(1-I_1) \\ -V \end{bmatrix}$$

$$= \begin{bmatrix} -18+20I_1 \\ -V \end{bmatrix} = \begin{bmatrix} -18 \\ -V \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} R-8 & -R \\ -R & R+4 \end{bmatrix}\begin{bmatrix} I_1 \\ I \end{bmatrix} = \begin{bmatrix} -18 \\ -V \end{bmatrix}.$$

By Cramer rule,

$$I = \frac{\begin{vmatrix} R-8 & -18 \\ -R & -V \end{vmatrix}}{\begin{vmatrix} R-8 & -R \\ -R & R+4 \end{vmatrix}} = \frac{R-8}{4(R+8)}V + \frac{18R}{4(R+8)}. \tag{48}$$

Figure 20*(d)* gives the circuit after replacing the circuit to the left of XY by its Thevenin equivalent. Now

$$I = -\frac{1}{R_T}V + \frac{V_T}{R_T}. \tag{49}$$

Comparing eqns (48) and (49) we get

$$R_T = -\frac{4(R+8)}{R-8},\ V_T = -\frac{18\,R}{(R-8)} \tag{50}$$

Now P is given by

$$P = \left[\frac{\dfrac{-18R}{R-8}}{\dfrac{-4(R+8)}{R-8} + R_L}\right]^2 R_L = \frac{[-18R]^2 R_L}{[(R_L-4)R - 8(R_L+4)]^2} = \frac{V_T{'}^2 R_L}{[R_T{'} - 8(R_L+4)]^2} \tag{51}$$

where $V_T{'} = -18R,\ R_T{'} = -(R_L+4)$ both are the polynomials in R. The condition for maximum/minimum power, from eqn. (39), is

$$\frac{V_T{'}}{R_T{'} - 8(R_L+4)} = \frac{\dfrac{d}{dR}V_T{'}}{\dfrac{d}{dR}R_T{'}}$$

$$\frac{-18R}{(R_L-4)R - 8(R_L+4)} = \frac{\dfrac{d}{dR}\{-18R\}}{\dfrac{d}{dR}\{(R_L-4)R\}} = \frac{-18}{R_L-4}.$$

This does not yield any finite value for R. This implies that P is a monotonic function. From eqn. (51), we see that P increases as R increases. Hence it will be minimum when $R = 0$ and maximum when $R = \infty$. Thus the maximum power is

$$\frac{324}{(R_L-4)^2} = 36 \text{ when } R_L = 1.$$

V_T and R_T are given by eqn.(50).

B. External current source and loop analysis

External current source is applied across XY as shown in Figure 21*(a)*. Converting the current sources into voltage sources, we get the circuit shown in Fig. 21(*b*).

By KVL

$$(R+12)I_1 = 2 - 20 I_X - RI$$

$$\Rightarrow I_1 = -\frac{RI+18}{R-8}.$$

Now

$$V = RI_1 + (R+4)I = -\frac{4(R+8)}{R-8} I - \frac{18R}{R-8}. \qquad (52)$$

However, from eqn. (3), (here current I is negative)

$$V = R_T I + V_T. \qquad (53)$$

Comparing this with eqn. (49), we get the same values of the V_T and R_T given by eqn.(50).

We shall take a general case of Examples 3 and 4. From eqn. (53),

$$I = -\frac{1}{R_T} V + \frac{V_T}{R_T} \qquad (54)$$

In general $I = aV + b$. Thus, comparing we get

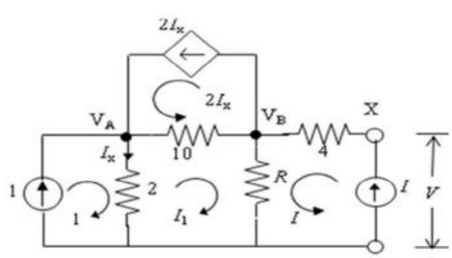
$$R_T = -1/a, V_T = -b/a.$$

It is obvious that both R_T and V_T have the same denominator a. Thus we may express

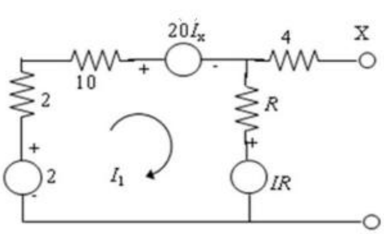
$$V_T = \frac{uR+v}{wR+x}, \qquad R_T = \frac{yR+z}{wR+x}. \qquad (55)$$

where *u, v, w, x, y* and *z* are constants.

Now



(a)



(b)

Figure 21 (*a*) Current source applied
(*b*) Current sources converted into voltage sources.

$$P(R) = \frac{[V_T]^2 R_L}{[R_T + R_L]^2} = \frac{\left[\frac{uR+v}{wR+x}\right]^2 R_L}{\left[\frac{yR+z}{wR+x} + R_L\right]^2}$$

$$= \frac{[uR+v]^2 R_L}{[(y+wR_L)R + (z+xR_L)]^2}. \qquad (56)$$

This will be maximum/minimum when, from eqn.(39),

$$\frac{[uR+v]^2 R_L}{[(y+wR_L)R+(z+xR_L)]^2} = \frac{2[uR+v]R_L}{2[(y+wR_L)R+(z+xR_L)](y+wR_L)}. \qquad (57)$$

It can be seen that this does not yield the value of R. Thus P is a monotonic function. Its value at $R = 0$ and $R = \infty$ are respectively

$$P(0) = \frac{v^2 R_L}{[z+xR_L]^2}$$

$$P(\infty) = \frac{u^2 R_L}{[y+wR_L]^2}. \qquad (58)$$

Thus P will increase from $P(0)$ to $P(\infty)$ if $P(0) < P(\infty)$, i.e., when

$$\frac{v^2 R_L}{[z+xR_L]^2} < \frac{u^2 R_L}{[y+wR_L]^2},$$

i.e., when

$$\left[R_L - \left(\frac{uz - vy}{wv - ux}\right)\right]\left[R_L + \left(\frac{uz+vy}{wv+ux}\right)\right] \geq 0$$

$$R_L \geq \left(\frac{uz-vy}{wv-ux}\right) \text{ or } -\left(\frac{uz+vy}{wv+ux}\right) \qquad (59)$$

From eqns.(58) and (59), we observe that

$$R_L \geq 0, \qquad P(0) \leq P(\infty)$$

$$0 \geq R_L \geq \left(\frac{uz-vy}{wv-ux}\right), \qquad P(0) \geq P(\infty)$$

$$\left(\frac{uz-vy}{wv-ux}\right) \geq R_L \geq -\left(\frac{uz+vy}{wv+ux}\right), \quad P(0) \leq P(\infty) \qquad (60)$$

$$-\left(\frac{uz+vy}{wv+ux}\right) \geq R_L \qquad P(0) \geq P(\infty)$$

From eqn.(60), we note that

1. The maximum/minimum power occurs when either $R = 0$ or $R = \infty$, *i.e.*, when either R is short circuited or open circuited.
2. For positive values of R_L, $P(R) > 0$ and vice versa.
3. When $u = v$, $w = x$, $y = z$, $P(0) = P(\infty)$. In this case both V_T and R_T become independent of R. It is also possible that $P(0)$ may be equal to $P(\infty)$ for some value of R_L. In both the cases, the power will be constant (independent of R). Readers may find some interesting application of this property.

The above results are summarized in the form of the following theorem.

Theorem: In Figure 1, if Thevenin components

$$R_T = \left(\frac{yR+z}{wR+x}\right) \text{ and } V_T = \left(\frac{uR+v}{wR+x}\right),$$ then the power in

R_L increases with increase in R for $R_L > 0$ and for

$$\left(\frac{uz-vy}{wv-ux}\right) > R_L > -\left(\frac{uz+vy}{wv+ux}\right),$$ otherwise decreases,

and remains constant when

$$R_L = \left(\frac{uz-vy}{wv-ux}\right) \text{and} -\left(\frac{uz+vy}{wv+ux}\right).$$

In example 3, we have $u = v = 10$, $w = 1$, $x = y = 2$, $z = 3$.

From eqn. (58) $P(0) = \dfrac{100\,R_L}{[3+2R_L]^2}$ and $P(\infty) = \dfrac{100\,R_L}{[2+R_L]^2}$.

$P(0)$ has zeros at $R_L = 0$ and ∞, two

poles at $R_L = -3/2$ and peak at $R_L = 1.5$; while $P(\infty)$ has zeros at $R_L = 0$ and ∞, poles at $R_L = -2$ and peak at $R_L = 2$. From these values, the plots of both $P(0)$ and $P(\infty)$ versus R_L are drawn in Fig.22.

It can be seen that the maximum power (infinite) is delivered by $R_L = -1.5$ when R is shorted or by $R_L = -2$ when R is opened. However, the maximum power 12.5 is delivered to R_L when its value is 2 and R is opened.

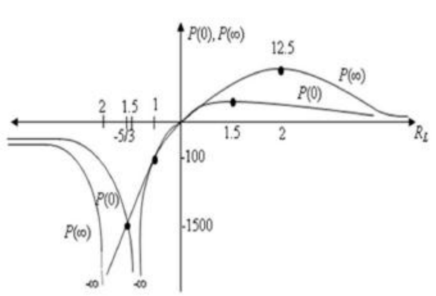


Figure 22. Variation of $P(0)$ and $P(\infty)$ versus R_L for the circuit of Example 3.

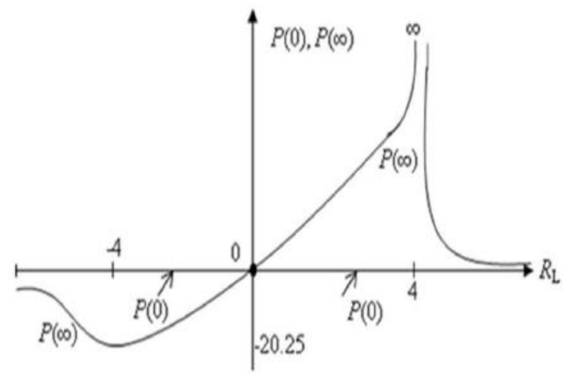


Figure 23. Variation of $P(0)$ and $P(\infty)$ versus R_L for the circuit of Example 4.

Since $P(0)$ and $P(\infty)$ curves intersect, other than at the trivial cases $R_L = 0, \infty$, at $R_L = -1, -5/3$ giving $P(0) = P(\infty)$. Hence, this circuit delivers constant power –100 for $R_L = -1$ and -1500 for $R_L = -5/3$.

In Example 4, $u = -18$, $v = 0$, $w = 1$, $x = -8$, $y = -4$, $z = -32$.

From eqn (58) $P(0) = 0$ and $P(\infty) = \dfrac{324\,R_L}{[-4+R_L]^2}$

$P(0)$ has zeros at $R_L = 0$ and ∞; while $P(\infty)$ has zeros at $R_L = 0$ and ∞, two poles at $R_L = 4$ and a peak at $R_L = -4$. From these values, the plots of both $P(0)$ and $P(\infty)$ versus R_L are drawn in Fig. 10. It can be seen that the maximum power (infinite) is delivered to R_L when its value is 4. Since $P(0)$ and $P(\infty)$ curves do not intersect, except at the trivial cases $R_L = 0, \infty$, $P(0) \neq P(\infty)$ for $R_L \neq 0, \infty$. Hence, this circuit does not deliver constant power for any value of R_L.

III. GENERALIZED MAXIMUM POWER DELIVERY THEOREM

Consider the circuit shown in Fig. 24 where 2-terminal network n is delivering the power to 2-terminal network N. Network n is shown by its Thevenin equivalent where v_T is Thevenin voltage equal to open circuit voltage v_{oc} and R_T is the Thevenin resistance = open circuit voltage v_{oc}/short circuit current i_{sc}. KVL gives

$$v = -R_T i + v_T = -R_T i + v_o \qquad (61)$$

Plot of eqn. (61) is shown in Fig. 26. Note that it has a negative slope being power delivery. The interconnection of networks n and N is across terminals AB. Hence both will have the same voltage v and current i. The v-i characteristics of N will have positive slope to receive power. It is therefore necessary that v-i characteristic 2 of N should intersect at a point p as shown in Fig. 26. The power delivered by n to N is given by

$$P = vi. \qquad (62)$$

After substituting for v from eqn. (61), the condition for maximum p, mentioned in the theorem, is obtained in the usual manner using differentiation etc.

An alternative proof [9] is given below.

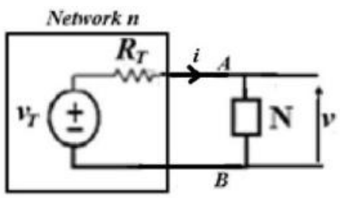


Figure 24. Circuit with network n replaced by Thevenin Equivalent and 2-termnal network N connected.

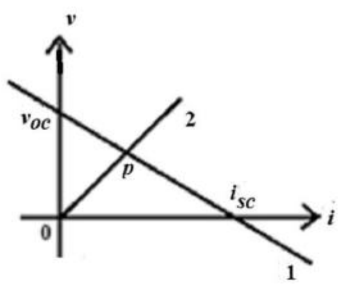


Figure 25. v-i characteristics of n (1) and N (2).

Alternative Proof of the Generalized Maximum Power Transfer Theorem

In Fig. 25 N, as per substitution theorem, is replaced by a voltage source of value v as shown in Fig. 26. Now current i can be expressed as

$$i = \frac{v_T}{R_T} - \frac{v}{R_T} \tag{63}$$

Substituting for $R_T = v_{oc}/i_{sc}$ and $v_T = v_{oc}$

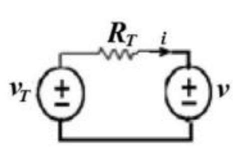


Figure 26. Circuit of Fig 24 with 2-termnal network N replaced by a voltage source v.

$$i = i_{sc} - \frac{i_{sc}}{v_{oc}} v. \tag{64}$$

Now power absorbed by the N is

$$p = vi = i_{sc} v - \frac{i_{sc}}{v_{oc}} v^2.$$

Power p will be maximum when

$$\frac{dp}{dv} = 0 \Rightarrow i_{sc} - 2\frac{i_{sc}}{v_{oc}} v = 0$$
$$\Rightarrow v = \frac{1}{2} v_{oc} \tag{65}$$

Now from eqn. (64)

$$i = i_{sc} - \frac{i_{sc}}{v_{oc}} v = \frac{1}{2} i_{sc}. \tag{66}$$

Results of eqns. (65) and (66) can be stated in the following generalized maximum power transfer theorem.

Generalized maximum power transfer theorem: *The maximum power will be transferred from a 2-terminal network n to another 2-terminal network N when voltage across (current through) N equals half of open circuit voltage (half of short circuit current) of network n.*

The above theorem can also be proved by connecting a current source i instead of v.

The essential difference in the present derivation and that due to Narayanan is that instead of keeping the network N intact, we replace it by either a voltage source v (or a current source i) for finding power p which is more convincing. Note that the theorem gives the values of voltage and current for maximum power which are solely decided by network n. The actual device which will absorb the maximum power will be decided by the v-i characteristic of the network N.

Example 5: Find the conditions when the 2-terminal network N shown in Fig. 27 receives the maximum power under the following cases: N is (a) a current source, (b) a voltage source (c) resistance (d) a series resistance R and a voltage source of 7.5 V (e) a nonlinear device which has v-i relation $v = ki^2$.

Here v_{oc} = 30 V and i_{sc} = 30/2 = 15 A. For maximum power to be delivered to N, voltage across it should $v_{oc}/2$ = 30/2 =15 V or current should be $i_{sc}/2$ = 15/2 = 7.5 A.

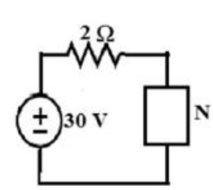


Figure 27. Given circuit.

Current source: Current through N for maximum power should be $i_{sc}/2$ = 7.5 A.

a *Voltage source:* Voltage across the voltage-source for maximum power should be $v_{oc}/2 = 15$ V.

b *Resistance:* When a resistance is connected, the voltage across this for maximum power to be delivered should be $v_{oc}/2 = 15$ V and current through it should be $i_{sc}/2 = 7.5$ A. Thus the resistance should be 15/7.5 = 2 which is the same given by the traditional maximum power transfer theorem.

c *Series combination of voltage and resistance:* For maximum power, the voltage across the combination should be $v_{oc}/2 =$ 15 V and current through it should be $i_{sc}/2 = 7.5$ A. Voltage across the resistance is 15 – 7.5 = 7.5 V. Therefore the resistance should be, by Ohm's law, 7.5/7.5 = 1 Note that the value of R equal to 1 gives the maximum power in the complete series combination of R and the 7.5–V voltage source and not in the R alone. For maximum power in R alone, its value should be

$(v_{oc}/2)/(i_{sc}/2) = [(30 – 7.5)/2]/[(30–7.5)/4] = 2$.

$v_{oc}/2 = k(i_{sc}/2)^2 \rightarrow 15 = k(7.5)^2 \rightarrow k = 4/15$.

IV. SUPERPOSITION THEOREM

Analysis of circuits with controlled sources using Principle of Supposition POS : We prove that POS can be applied in 'true sense' in solving the circuits with controlled sources. Here 'true sense' means that the response due to all the independent and dependent sources is obtained by superimposing the responses obtained, considering one source at a time. For convenience, without any loss of generality, we take the typical two-node network shown in Fig. 28, with current sources only as it is explained in that voltage sources, if present, can be converted into current sources. Using node analysis, one can write

$$\begin{bmatrix} y_{11} & -y_{12} \\ -y_{12} & y_{22} \end{bmatrix}\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \sum I_x \\ \sum I_y \end{bmatrix} = \begin{bmatrix} I \\ -I + kV_x \end{bmatrix}. \qquad (67)$$

Note that $\sum I_x$ and $\sum I_y$ may or may not contain the independent and/or dependent sources depending upon the position of the current sources in the circuit. In the circuit shown, node X has the independent current source I only while node Y has both the independent current source I and dependent current source kV_x. Eqn. (67) can be rewritten as

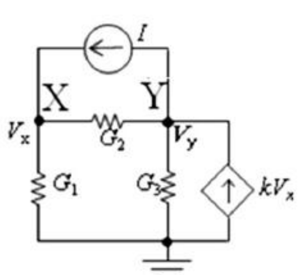


Figure 28. Typical 2-node network.

$$\begin{bmatrix} y_{11} & -y_{12} \\ -y_{12} & y_{22} \end{bmatrix}\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} I \\ -I \end{bmatrix} + \begin{bmatrix} 0 \\ kV_x \end{bmatrix} \qquad (68)$$

$$= R_i + R_d$$

where R_i is the response due to the independent source I and R_d is that due to the dependent source kV_x. It is obvious from eqn (68) that the node voltages can be solved by the POS. For example

$$V_x = V_{x1} + V_{x2}$$

$$V_x = \frac{\begin{bmatrix} I & -y_{12} \\ -I & y_{22} \end{bmatrix}}{\Delta} + \frac{\begin{bmatrix} 0 & -y_{12} \\ kV_x & y_{22} \end{bmatrix}}{\Delta} \qquad (69)$$

where $\Delta = y_{11}y_{22} - y_{12}^2$.

Here dependent source kV_x should be treated as an independent source of value kV_x where V_x is the full and final value, *i.e.*, when all the sources (independent and dependent) are present. Hence, it can be deactivated without reducing the controlling variable V_x to zero while determining the response due to the independent source I, like we do not put any current through, or voltage across, any element 0 while deactivating an independent source.

Solving for V_x from eqn. (69), one gets

$$V_x = \frac{(y_{22} - y_{12})}{\Delta - y_{12}k} I \qquad (70)$$

Similarly, from eqn. (68), by Cramer's rule, one gets

$$V_y = \frac{\begin{bmatrix} y_{11} & I \\ -y_{12} & -I \end{bmatrix}}{\Delta} + \frac{\begin{bmatrix} y_{11} & 0 \\ -y_{12} & kV_x \end{bmatrix}}{\Delta}$$

$$= \frac{-y_{11} + y_{12}}{\Delta} I + \frac{y_{12}}{\Delta} kV_x.$$

Substituting for V_x from (70), and simplifying

$$V_y = \frac{(-y_{11} + y_{12} + k)}{\Delta - y_{12}k} I. \qquad (71)$$

Now we solve the circuit by the matrix method of [23]. Equation (68) can be expressed as

$$\begin{bmatrix} y_{11} & -y_{12} \\ -y_{12} & y_{22} \end{bmatrix}\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} I \\ -I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix}\begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

which yields

$$\begin{bmatrix} y_{11} & -y_{12} \\ -y_{12} - k & y_{22} \end{bmatrix}\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} I \\ -I \end{bmatrix}.$$

On solving one gets

$$V_x = \frac{(y_{22} - y_{12})}{\Delta - y_{12}k} I \quad (72)$$

and

$$V_y = \frac{(-y_{11} + y_{12} + k)}{\Delta - y_{12}k} I \quad (73)$$

Equations (72) and (73) are the same as eqns (70) and (71), respectively. Thus, we conclude that POS can be applied to linear circuits with controlled sources also.

In [11], it is mentioned that POS cannot be applied to networks when all the sources but one are deactivated and *the resulting circuit contains a node at which the voltage is indeterminate or a branch in which the current is indeterminate*. In such cases POS cannot be used even if all sources are independent. We state this condition in a more general form. POS cannot be applied to circuits with or without independent sources when all the sources but one are deactivated, the activated source should not become open if it is a current source or short if it is a voltage source. Two examples of such circuits are shown in Figs. 29(*a*) and (*b*) where the current source is opened and the voltage source is shorted, respectively. The circuit in Fig. 1 is solvable when one of the current sources, say I_3 is a dependent source such that $I_3 = I_1 + I_2$ (requirement of KCL). If we further make that $I_3 = gV$, the circuit becomes unsolvable because two constraints on I_3 cannot simultaneously be satisfied. Similarly, the circuit shown in Fig. 29(*b*) is solvable when one of the voltage sources, say V_{CA}, is a dependent source such that $V_{CA} = -(V_{AB} + V_{BC})$ (requirement of KVL), but becomes unsolvable when V_{CA} is also dependent on some other voltage or current in the circuit.

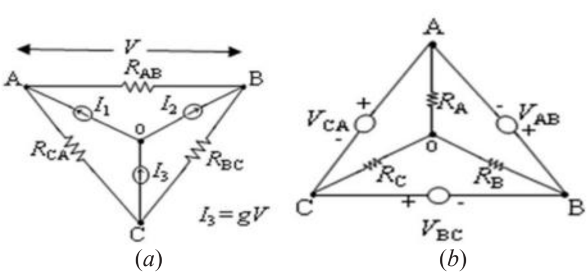


Figure 29. Circuits not solvable by POS.

Example 6: Determine the output voltage V_o in the circuit shown in Fig. 30.

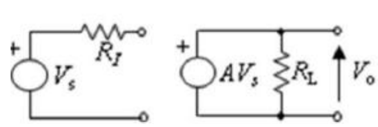


Figure 30. Circuit for Example 1.

Applying POS

$V_o = V_1$ (due to the source V_s alone) + V_2 (due to thesource AV_s alone) = AV_s +0=AV_s.

Example 7: Find the current through G_2 in Fig. 28 when G_1 = 0.8 S, G_2 = 0.2 S, G_3 = 0.3 S, k = 0.8 S, I = 23 A. Applying POS

$$V_x = I\frac{G_a}{(G_a + G_2)G_1} - kV_x\frac{G_b}{(G_b + G_3)G_1}$$

where

$$G_a = \frac{G_1G_3}{G_1 + G_3}, \quad G_b = \frac{G_1G_2}{G_1 + G_2}$$

Substituting the values, one gets

$$V_x = 15 + (8/23)V_x \;\rightarrow\; V_x = 23\,\mathrm{V}.$$

By POS for V_y

$$V_y = I\frac{-G_a}{(G_a + G_2)G_3} + kV_x\frac{1}{(G_b + G_3)}$$
$$= -40 + (40/23)V_x = -40 + 40 = 0.$$

Note that it is easier to solve for the controlling variable V_x by POS first and then any other voltage or current, if required, by any other method including using POS.

Example 8: Determine current I in the circuit shown in Fig. 31.

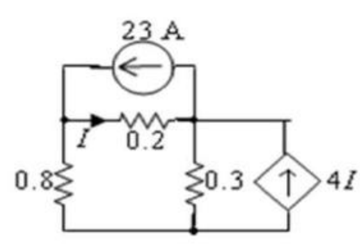


Figure 31. Circuit for Example 8.

By POS,

$$I = 23\frac{G_a}{(G_a + G_2)} - 4I\frac{G_b}{(G_b + G_3)}$$
$$= 11 - (32/23)I \;\rightarrow I = 4.6\ \mathrm{A}.$$

Example 9: Consider the circuit shown in Fig. 32.
This circuit cannot be solved by series-parallel reduction, current and voltage division and Ohm's law. We solve it by matrix method [33].

By POS and using node analysis, one gets

$$V=\frac{\begin{bmatrix}3 & 1 & -1\\ -1 & 0 & -1\\ -1 & 0 & 3\end{bmatrix}}{\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}}+\frac{\begin{bmatrix}3 & 0 & -1\\ -1 & 0 & -1\\ -1 & 2V & 3\end{bmatrix}}{\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}}=\frac{1}{4}+\frac{1}{2}V$$

$$\Rightarrow V=\frac{1}{2}\ \text{V}.$$

This is the correct answer verified by other method.

If a network does not have a single independent source, but has dependent sources only, then from eqn. (68), we see that $R_i = 0$ and consequently, R_d will also be zero. It means that, in the absence of any independent source, the circuit is dead, *i.e.*, no current through, and voltage across, any element exist, even though the dependent source(s) may be present.

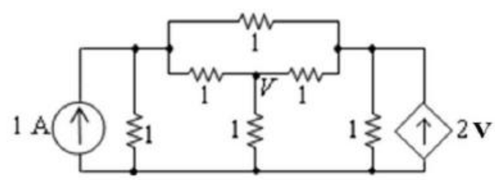
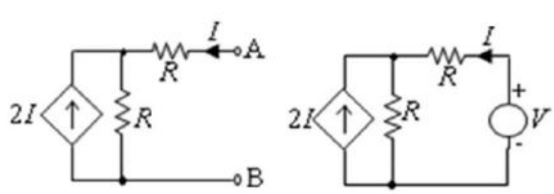


Figure 32. Circuit for Example 4.

While determining Thevenin equivalent of a circuit without any independent source but with dependent source, both the open circuit voltage and the short circuit current will be zero as explained above. In such a case, Thevenin resistance would be indeterminate using the relation $R_{th} = V_{oc}/I_{sc}$ =0/0. However, if we connect an independent voltage (current) source of value V (I) at the output terminalsand find the current I flowing into the voltage source (voltage drop V developed across the current source), then $R_{th} = V/I$.

Example 10: Find the Thevenin equivalent of the circuit across the terminals AB shown in Fig. 33(*a*).

We connect a voltage source at the terminals AB. By POS



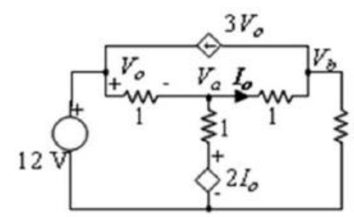
(*a*) (*b*)

Figure 33. (*a*) Circuit for Example 10, (*b*) Circuit with external voltage source connected.

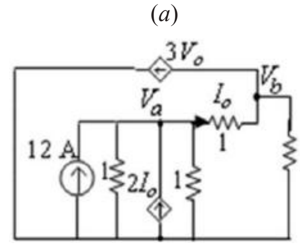
$$I=\left(\frac{1}{2R}\right)V-I \;\rightarrow\; \frac{V}{I}=R_{Th}=4R.$$

Example 11: Determine the node voltages V_a and V_b in the circuit shown in Fig. 34(*a*). There are two dependent sources; one is controlled by a voltage V_o and the other by current I_o which require the evaluation of corresponding difference of two node voltages. Such controlling variables almost double the complexity of the solution by POS.

We apply the node analysis. By POS, one gets



(*a*)



(*b*)

Figure 34. (*a*) Circuit for Example 6 and (*b*) reduced circuit

$$V_a=\frac{\begin{vmatrix}12 & -1\\ 0 & 2\end{vmatrix}+\begin{vmatrix}2I_o & -1\\ 0 & 2\end{vmatrix}+\begin{vmatrix}0 & -1\\ -3V_o & 2\end{vmatrix}}{\begin{vmatrix}3 & -1\\ -1 & 2\end{vmatrix}} \tag{74}$$

$$=\frac{24+4I_o-3V_o}{5}$$

$$V_b=\frac{\begin{vmatrix}3 & 12\\ -1 & 0\end{vmatrix}+\begin{vmatrix}3 & 2I_o\\ -1 & 0\end{vmatrix}+\begin{vmatrix}3 & 0\\ -1 & -3V_o\end{vmatrix}}{\begin{vmatrix}3 & -1\\ -1 & 2\end{vmatrix}} \tag{75}$$

$$=\frac{12+2I_o-9V_o}{5}$$

$$\text{Now}\quad I_o=(V_a-V_b)/1$$

$$=\frac{24+4I_o-3V_o}{5}-\frac{12+2I_o-9V_o}{5} \tag{76}$$

$$\Rightarrow 2V_O - I_O = -4$$

$$\text{Now} \quad V_O = 12 - V_a = 12 - \frac{24 + 4I_O - 3V_O}{5} \tag{77}$$

From eqns. (76) and (77), by Cramer's rule

$$V_O = \frac{\begin{vmatrix} -4 & -1 \\ 18 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}} = 2 \text{ V}, \quad I_O = \frac{\begin{vmatrix} 2 & -4 \\ 1 & 18 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}} = 8 \text{ A}.$$

Substituting the values of V_o and I_o in eqns. (76) and (77) one gets

$$V_a = 10\text{V}, \ V_b = 2\text{V}.$$

Now let us solve the same problem by Matrix method [33]. Node analysis gives

$$\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 12 + 2I_O \\ -3V_O \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 2(V_a - V_b) \\ -3(12 - V_a) \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 2V_a - 2V_b \\ -36 + 3V_a \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ -36 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 12 \\ -36 \end{bmatrix}$$

By Cramer's rule

$$V_a = \frac{\begin{vmatrix} 12 & 1 \\ -36 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}} = 10 \text{ V}, \quad V_b = \frac{\begin{vmatrix} 1 & 12 \\ -4 & -36 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}} = 2 \text{ V}$$

These are the same as obtained above, but with considerably less effort in solving.

Comparison with Other Methods

There is a similarity between the methods based on POS and Miller equivalents [34]. In the former method, the sources are dependent while in the latter method, the elements are dependent on some parameter. However, in both the methods, one has to determine the controlling variables first and then any other desired voltage or current. As proved in [34], matrix method is more efficient than the Miller equivalent approach. It is also more efficient than the method based on POS. This is proved below.

Let there be N number of unknown nodes and S_i and S_d be the number of independent and dependent sources, respectively, in a circuit. We shall compare the number of determinants to be solved by the POS method and the matrix method for determining the voltages of N nodes. In POS method, N equations for N node voltages in terms of controlling variables are to be written invoking POS. These relations require $N(S_i + S_d) + 1$ determinants of the order $N \times N$ to be solved. After this S_d relations among the controlling variables will be determined. Then evaluation of the controlling variables from these relations requires $S_d + 1$ determinants of order $S_d \times S_d$ to be solved. After this the voltages of N unknown nodes are evaluated. Thus in the above example, since $N = 2$, $S_i = 1$ and $S_d = 2$, it requires 10 determinants of order 2×2 to be solved.

Matrix method requires only $N + 1$ determinants of order $N \times N$ to be solved. Thus, it requires only 3 determinants as against 10 by POS for the circuit of example 6. There is no need to determine the controlling variables explicitly. Thus the matrix method is more efficient, easier and straight forward.

V. CONCLUSION

Three possible cases of maximum power transferred to a load in a circuit where only one resistance is variable, have been brought out. Only in of the cases when the *load resistance is variable* the maximum power transfer theorem can be applied. In other two cases, *when the load resistance R_L is fixed*, power has to be calculated from the first principle. Conditions for maximum power transfer have been derived for these cases. A case when both the Thevenin Voltage and Thevenin Resistance are variable has been studied in detail. The maximum/minimum power is obtained when either $R = 0$ (short circuit) or ∞ (open circuit). When R_L is positive it absorbs the power; while negative it delivers the power to the circuit. Either power can be infinite, constant irrespective of the variable resistance, and finite for specific values of R_L. This result has been stated in the form of a theorem. Theory has been explained with the help of typical examples.

A generalized maximum power theorem has been stated and proved by two different methods.

In the text books, while solving the circuits with controlled source using POS, controlled sources are not deactivated. Thus POS has not been applied in 'true sense' to circuits with dependent sources. It has been shown here that POS can be applied in the 'true sense' to such circuits also, but with the following caution: (*i*) All the dependent sources should also be treated as independent sources with their full value (contribution from all the sources). (*ii*) When the dependent source is deactivated, its controlling variable should not be zeroed. POS is applicable to all those circuits with dependent sources as well if it is applicable to these circuits when all the dependent sources are treated as independent sources.

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